Time-stepping for multibody dynamics with friction-affected bilateral constraints

Li Fu\textsuperscript{a,b,*}, Qi Wang\textsuperscript{a}, Shimin Wang\textsuperscript{a}

\textsuperscript{a} Department of dynamics and control, Behang University, Beijing 100191, China
\textsuperscript{b} Science School, Hebei Polytechnic University, Tangshan 063000, China

Received 20 October 2008; received in revised form 11 May 2009; accepted 31 May 2009

Abstract

The dynamics of multibody systems with friction-affected bilateral constraints is essentially different from those of smooth multibody systems. In this paper, general modeling and numerical methods for this kind of friction-affected system are given. Each friction-affected bilateral constraint is modeled by splitting it into two unilateral constraints opposite to each other. The constraint equalities are replaced with complementarity inequalities to avoid the absolute value terms in the dynamic equations. A linear complementarity problem time-stepping scheme is presented for simulation that does not suffer from the difficulties of enumeration (known as Delassus’ problem). The algorithm has the advantage that it needs no detection for stick-slip transition when neglecting the difference between the static and kinetic friction coefficients. Our method is carried out in an example to analyze the nonsmooth nonlinear behavior of a typical mechanism.

© 2009 National Natural Science Foundation of China and Chinese Academy of Sciences. Published by Elsevier Limited and Science in China Press. All rights reserved.

Keywords: Friction; Multibody dynamics; Unilateral constraints; Time-stepping scheme

1. Introduction

Present research in the field of nonsmooth mechanics mainly involves unilaterally constrained mechanical systems [1]. Multibody systems with friction-affected bilateral constraints exist widely in practical engineering, but they are rarely given special attention. The present paper studies this kind of mechanical system, which has a finite number of friction-affected translational planar joints, but neglects the friction in revolute joints. The aim of this paper is to present a general method for modeling and numerically simulating the multibody systems of this type.

Our system exhibits the features of a nonsmooth mechanical system. Due to the set-value mapping characteristic of dry friction forces and the swichovers of constraints, the global motion of the system can be described with many different states, and the differential equations of motion have discontinuous right-hand vector fields. By neglecting the clearance and the effect of impact between rigid bodies and constraints, the state variables in the differential equations are continuous, therefore allowing our system to be classified as a Filippov system [2].

References [3–8] used the piecewise differential algebra equation (DAE) approach for modeling the dynamics of this kind of system. The friction-affected bilateral constraints are considered as equations. Because of Coulomb’s friction law, Lagrange multipliers with absolute value terms appear in the dynamic equations of the system. To numerically solve a DAE with absolute value terms, an enumeration method can be used to detect the signs of the Lagrange multipliers [3–5], but the calculations increase exponentially with the growth in the number of friction-affected constraints. Another major obstacle is the nonexistence and nonuniqueness of solution caused by the absolute value terms. The mechanical phenomena associated with the nonexistence

Accepted Manuscript
and nonuniqueness of solution under certain conditions has been discussed in Refs. [6–8]. The linear complementarity problem (LCP) modeling method based on the decomposition of normal forces was suggested in Ref. [9].

The modeling method we propose is based on constraint decomposition. Bilateral constraints are substituted with unilateral constraints, constraint equalities are substituted with complementarity conditions, and the DAE with absolute value terms are substituted with structure-varying dynamic equations. A time-stepping method is adopted for the computational algorithm. Compared with an event-driven scheme, the time-stepping method has the advantage that it needs no detection of stick-slip transition when neglecting the difference between the static and kinetic friction coefficients [10]. This method is especially useful when the system has many friction-affected constraints and frequent constraint switchovers.

2. Complementarity conditions and kinetic equations

2.1. Constraint decomposition

We assume a total of \( m \) friction-affected bilateral constraints in a planar multibody system, with each one having both normal geometric constraints and tangential friction constraints.

The normal constraints are bilateral and described by the equation \( g_{Ni} = 0 \), \( i \in \{1, 2, \ldots, m\} \). Here, \( g_{Ni} \) is the normal constraint function. The absolute value Lagrange multipliers are avoided in the dynamics equations by separating the normal constraints into a pair of unilateral constraints opposite to each other. The normal constraint functions \( g_{Ni} \) and their corresponding Lagrange multipliers \( \lambda_{Ni} \) must therefore be decomposed into positive and negative parts.

The normal constraint functions are decomposed with:

\[
\begin{align*}
g_{Ni}^+ & = (|g_{Ni}| + g_{Ni})/2 \\
g_{Ni}^- & = (|g_{Ni}| - g_{Ni})/2 \\
g_{Ni} & = g_{Ni}^- - g_{Ni}^+ 
\end{align*}
\]

and the normal reaction forces are decomposed with:

\[
\begin{align*}
\lambda_{Ni}^+ & = (|\lambda_{Ni}| + \lambda_{Ni})/2 \\
\lambda_{Ni}^- & = (|\lambda_{Ni}| - \lambda_{Ni})/2 \\
\lambda_{Ni} & = \lambda_{Ni}^- - \lambda_{Ni}^+
\end{align*}
\]

The normal constraint functions \( g_{Ni}^+ \) and \( g_{Ni}^- \) are complementary to the Lagrange multipliers \( \lambda_{Ni}^+ \) and \( \lambda_{Ni}^- \) as follows:

\[
\begin{align*}
\lambda_{Ni}^+ & \geq 0, \quad g_{Ni}^+ \leq 0, \quad g_{Ni}^+ \lambda_{Ni}^+ = 0 \\
\lambda_{Ni}^- & \geq 0, \quad g_{Ni}^- \geq 0, \quad g_{Ni}^- \lambda_{Ni}^- = 0
\end{align*}
\]

The above complementary conditions can also be expressed as follows:

\[
\begin{align*}
\lambda_{Ni}^+ & \geq 0, \quad g_{Ni}^+ \leq 0, \quad g_{Ni}^+ \lambda_{Ni}^+ = 0 \\
g_{Ni}^+ & = g_{Ni}^+
\end{align*}
\]

where \( \lambda_{Ni}^+ \) and \( g_{Ni}^+ \) are contained in the vectors \( \lambda_{Ni}^+ \) and, respectively.

The tangential Coulomb friction law can be expressed as the following two possible cases:

\[
\begin{align*}
\dot{g}_{T} & \neq 0, \quad \dot{\lambda}_{T} = -\mu |\dot{\lambda}_{N}| \text{sign}(\dot{g}_{T}) \\
\dot{g}_{T} & = 0, \quad \mu |\dot{\lambda}_{N}| - |\dot{\lambda}_{T}| \geq 0
\end{align*}
\]

where \( \dot{g}_{T} \) is the relative sliding velocity, \( \dot{\lambda}_{T} \) is the friction force, \( \dot{\lambda}_{N} \) is the normal contact force and \( \mu \) is the friction coefficient (the difference between the static and \( \omega \) is the kinetic friction coefficients being negligible). When the difference between the static and the kinetic friction coefficients has to be taken into account, the even-driven scheme should be adopted, because the exact time of stick-slip transition must be detected.

The tangential friction law can also be decomposed into two separate unilateral complementarity conditions. First, we need to define the friction saturations [11], \( \lambda_{T}^+ \) and \( \lambda_{T}^- \), i.e. the distances within the friction cone from the end point to the cone surface in the positive and negative directions (see Fig. 2):

\[
\begin{align*}
\lambda_{T}^+ & = \mu |\dot{\lambda}_{N}| + \dot{\lambda}_{T} \\
\lambda_{T}^- & = \mu |\dot{\lambda}_{N}| - \dot{\lambda}_{T}
\end{align*}
\]

Adding the two equations above gives the following relation:

\[
\lambda_{T}^+ = 2\mu |\dot{\lambda}_{N}| - \lambda_{T}^-
\]

This will be of use later.

Furthermore, we split the tangential velocity \( \dot{g}_{T} \) into positive and negative parts:

\[
\begin{align*}
\dot{g}_{T}^+ & = (|\dot{g}_{T}| + \dot{g}_{T})/2 \\
\dot{g}_{T}^- & = (|\dot{g}_{T}| - \dot{g}_{T})/2 \\
\dot{g}_{T} & = \dot{g}_{T}^+ - \dot{g}_{T}^-
\end{align*}
\]

Fig. 1. The normal contact law.

Fig. 2. The tangential friction law.
The positive and negative friction saturations, $\lambda_{T0}^+$ and $\lambda_{T0}^-$, are complementary to the positive and negative tangential velocities, $g_T^+$ and $g_T^-$, as follows:

\[
\begin{align*}
\dot{g}_T^+ &> 0, \quad \dot{\lambda}_{T0}^+ > 0 \quad \Rightarrow \lambda_T^+ \dot{\lambda}_{T0}^+ > 0 \\
\dot{g}_T^- &> 0, \quad \dot{\lambda}_{T0}^- > 0 \quad \Rightarrow \lambda_T^- \dot{\lambda}_{T0}^- > 0
\end{align*}
\] (3)

The above expressions can also be represented by one formula:

\[
\dot{g}_T^- > 0, \quad \dot{\lambda}_{T0}^- > 0 \quad \Rightarrow \dot{\lambda}_T^- \dot{\lambda}_{T0}^- > 0
\] (3)

The characteristic lines related to conditions (1) and (3) are given in Figs. 1 and 2, respectively.

2.2. Dynamics equations

The dynamics of a multibody system with unilateral constraints can be expressed for all time by the following equation of motion [12]:

\[
M \ddot{q} - h - W_N \lambda_N - W_T \lambda_T = 0
\] (4)

where $q \in \mathbb{R}^n$ is the vector containing the generalized coordinates of the basic system. The basic system is derived from the original system by removing the nonsmooth constraints. The mass matrix $M \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. The term $h$ in Eq. (4) can be written as:

\[
h = Q + \partial T / \partial q - \dot{M} \ddot{q}
\]

where $Q$ contains the generalized forces of the basic system and $T$ is the kinetic energy. Also from Eq. (4), and $\lambda_T$ are the vectors of normal and tangential contact forces. The constraint matrices $W_N \in \mathbb{R}^{m \times n}$ and $W_T \in \mathbb{R}^{m \times n}$ (where $m$ is the number of bilateral constraints with friction) are the Jacobian matrices of the normal and tangential constraints, respectively [12].

The dynamics of a multibody system with friction-affected bilateral constraints can be expressed for all time by the following equation of motion:

\[
M \ddot{q} - h - W_N^+ \lambda_N - W_T^+ \lambda_T = 0
\] (5)

The normal constraint matrix $W_N^+$ is time variant. The constraint pairs $g_N^+ > 0 \quad (i = 1, 2, \ldots, m)$ alternate over time, as do the corresponding columns in $W_N^+$ and the corresponding terms in $\dot{\lambda}_N^+ \dot{\lambda}_{N0}^+$. In the latter part of this paper, the superscript $^+$ is used to denote terms that switch signs in the direction of their motion.

Eq. (5), together with constraint complementary conditions (1) and (3), completely describe the dynamics of the system with friction-affected bilateral constraints.

3. Time-stepping method based on LCP

3.1. Time-stepping formulation

We first use an Euler discretization of Eq. (5), resulting in the following equation:

\[
M \Delta \dot{u} - h \Delta t - W_N^+ \Delta \lambda_N - W_T^+ \Delta \lambda_T = 0
\] (6)

\[
A \lambda_N = (u + \Delta u) \Delta t = 0
\] (7)

where $\lambda_N = \lambda_{N0} \Delta t$, and $\lambda_T = \lambda_{T0} \Delta t$.

Next, we discretize the unilateral constraints as follows [10]:

\[
\Delta g_N^+(q, t) = W_N^+ \Delta q + \dot{\lambda}_N \Delta t
\] (8)

\[
\dot{\lambda}_T = \dot{\lambda}_{T0} + \dot{\lambda}_T \Delta t
\] (9)

where $\Delta g_N^+(q, t) = (\partial g_N^+ / \partial q)^T$, $\dot{\lambda}_N = (\partial g_N^+ / \partial q)^T$, $\dot{\lambda}_T = d W_T / dt$, $\dot{\lambda}_{T0} = d \dot{\lambda}_{T0} / dt$, $\dot{\lambda}_T = d \dot{\lambda}_T / dt$, and $\dot{\lambda}_{T0} = d \dot{\lambda}_{T0} / dt$. The kinematic constraints at the end of a time step are defined with:

\[
g_N^+ := g_N^+ + \Delta g_N^+
\] (10)

\[
g_T^+ := \dot{g}_T + \Delta g_T^+
\] (11)

where the superscript $e$ denotes the end of the actual time step. We can put the contact laws from Eqs. (1) and (3) in the impulse form:

\[
g_N^+ \geq 0 \lambda_{N0} \geq 0 \Rightarrow \lambda_{N0}^+ \lambda_N^+ = 0
\] (12)

\[
\dot{g}_T^+ \geq 0 \lambda_{T0} \geq 0 \Rightarrow \lambda_{T0}^+ \dot{g}_T^+ = 0
\] (13)

where $\lambda_{T0}^+$ are the friction saturation impulses.

After eliminating $\Delta u$ and $\Delta q$ in Eqs. (8) and (9) using (6) and (7), then inserting $\lambda_N^+$ and $\dot{\lambda}_T^+$ into Eqs. (10) and (11), and introducing the impulse form of the contact laws finally in (5), the overall problem can be written as:

\[
\begin{bmatrix}
\lambda_{N0}^+ \\
\Delta \lambda_{T0}^+
\end{bmatrix}
= \begin{bmatrix}
G_{NN} - G_{NT} \mu & G_{NT} \\
G_{TN} - G_{TT} \mu & G_{TT}
\end{bmatrix}
\begin{bmatrix}
\Delta \lambda_{N0} \\
\Delta \lambda_{T0}^+
\end{bmatrix}
+ \begin{bmatrix}
W_N^+ \lambda_N \Delta t + g_N^+ \\
(G_T + h \Delta W_T^+ \dot{\lambda}_T + \dot{\lambda}_N^+ \Delta t + \dot{\lambda}_T^+)
\end{bmatrix}
\] (14)

where the following terms are defined to shorten the notation:

\[
G_{NN} = W_N^+ M^{-1} W_N^+, \quad G_{NT} = W_N^+ M^{-1} W_T,
\]

\[
G_{TN} = G_T W_N^+, \quad G_{TT} = G_T W_T,
\]

\[
G_T = (W_T^+ + \dot{W}_T) M^{-1}
\]

Expressions (12)–(14) constitute a LCP in standard form.

3.2. Integration procedure

- Input the initial conditions $q_0$ and $\dot{q}_0$, the intended final time $t_{end}$, the time step $\Delta t$, and the geometric and dynamic data of the problem.
- Choose any one contact state to obtain $W_N^+$ and $\dot{\lambda}_N^+$.

While $t < t_{end}$
• Find the normal constraint function \( g^*_N \).

If \( g^*_N > 0 \) (switchover of constraints)

Switch the signs of the corresponding columns in \( W^*_N \) and the corresponding terms in \( g^*_N, \dot{\omega}^*_N, \omega^*_N \), etc.

Else

\( W^*_N \) and \( \dot{\omega}^*_N \) defined above are the actual constraint matrices.

Endif

• Find \( \Lambda_N \) and \( \Lambda_T \), which are solutions of the LCP from Eqs. (12) to (14).

• Substitute Eq. (6) with \( \Lambda_N \) and \( \Lambda_T \) to find \( \Delta u \).

\( u_{\text{new}} = u + \Delta u; \quad q_{\text{new}} = q + u\Delta t; \quad \text{time} = \text{time} + \Delta t. \)

endwhile

4. Example

A slider-crank mechanism in a perpendicular plane is depicted in Fig. 3.

The mechanism has the following properties. Crank OA: mass \( m_1 \), length \( L_1 \); arm AB: mass \( m_2 \), length \( L_2 \); slider mass B: mass \( m_3 \). The torque applied to the crank is \( P = P_0 \sin(\omega t) \) (Nm). The linear elastic spring has stiffness \( k \), and is undeformed in the position \( x = -(R + L) \). The friction coefficient between the slider and the slide is \( \mu \). The acceleration of gravity is \( g = 9.8 \text{ m/s}^2 \). When the friction-affected bilateral constraint \( B \) is released, the system

![Fig. 4. Comparison of the displacement and velocity history with different frictional coefficients. (a) Displacement of \( \theta_1 \) (solid) and \( \theta_2 \) (dashed) at \( \mu = 0.3 \); (b) velocities of \( \theta_1 \) (solid) and \( \theta_2 \) (dashed) at \( \mu = 0.3 \); (c) displacement of \( \theta_1 \) (solid) and \( \theta_2 \) (dashed) at \( \mu = 0.5 \); and (d) velocities of \( \theta_1 \) (solid) and \( \theta_2 \) (dashed) at \( \mu = 0.5 \).](image-url)
has two degrees of freedom, which we gather in a vector of generalized coordinates \( \mathbf{q} = [\theta_1 \quad \theta_2]^T \). The normal contact distance and tangential contact velocity are:

\[
\mathbf{g}_N^* = \pm (L_1 \sin \theta_1 + L_2 \sin \theta_2) \\
\mathbf{g}_T = -L_1 \sin \theta_1 \cdot \dot{\theta}_1 - L_2 \sin \theta_2 \cdot \dot{\theta}_2
\]

which results in the following constraint matrices:

\[
W_N^* = \pm \begin{bmatrix} L_1 \cos \theta_1 \\ L_2 \cos \theta_2 \end{bmatrix}, \\
W_T = \begin{bmatrix} -L_1 & \sin \theta_1 \\ -L_2 & \sin \theta_2 \end{bmatrix}
\]

The equation of motion is written as:

\[
M \ddot{\mathbf{q}} = h + W_N^* \lambda_N + W_T \lambda_T
\]

(where \( W_N^* \) switches over between \( W_N^\pm \)).

The physical properties of the system are specified as follows: \( m_1 = 1 \text{ kg}, m_2 = 2 \text{ kg}, m_3 = 1 \text{ kg}, L_1 = 1 \text{ m}, L_2 = 2 \text{ m}, \)

\( k = 5 \text{ N/s}, P_0 = 10 \text{ Nm}, \) and \( \omega = \pi/6 \).

The initial conditions are specified as:

\( \theta_1(0) = \theta_2(0) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0 \).

4.1. Stick-slip motion

Fig. 4 depicts the nonsmooth dynamic behavior of the crank-slider mechanism with two different friction coefficients. Fig. 4(a) shows the time history of \( \theta_1 \) and \( \theta_2 \), which are the angular displacements of the crank and arm, respectively, with \( \mu = 0.3 \). Fig. 4(b) shows the time history of \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \), the corresponding angular velocities, with \( \mu = 0.3 \). In the above cases, no stick-slip motion appears. Fig. 4(c) and (d) is the time histories of the angular displacements and angular velocities, respectively, now with the friction coefficient set to \( \mu = 0.5 \). In the latter cases, obvious stick-slip motion appears.

Fig. 5 gives a comparison of the time-stepping scheme presented in this paper with the LCP-based event-driven scheme. Both algorithms adopt a fixed time step (\( \Delta t = 0.001 \text{ s} \)) to calculate the integral of the displacement \( \theta_1 \), with \( \mu = 0.5 \), over the time interval \([0, 30] \text{ s}\). The relative error curve between the two algorithms is shown in Fig. 5.

The results of the two algorithms show good agreement, with a maximum relative error of less than \( 1 \times 10^{-5} \text{ rad} \).

The qualitative change in stick-slip behavior for small changes in friction coefficient \( \mu \) can be summarized with a bifurcation diagram (Fig. 6). This plot shows the displacement \( \theta_1 \) of the crank at the stop as a function of the friction coefficients \( \mu \). No regions of sticking are visible.

Fig. 5. Comparison of the time-stepping algorithm and the acceleration-force event-driven scheme.

Fig. 7. Periodic solution and chaos. (a) \( \mu = 0.65, P_0 = 60 \); (b) \( \mu = 0.5, P_0 = 60 \); (c) \( \mu = 0.25, P_0 = 60 \).
for the range investigated and with $\mu < 0.05$. The number of stops is 2-4-6-4 during one period, as $\mu$ increases over the range $0.05 < \mu < 0.5$.

4.2. Periodic solution and chaos

Fig. 7 shows different types of phase plane plots for different frictional coefficients. The system exhibits one-periodic solutions (a), two-periodic solutions (b), and chaotic system behavior (c). For a more global examination of the bifurcation behavior of the system, representative points of the trajectories for each set of bifurcation parameters have been extracted in Fig. 8.

Particular features of the bifurcation diagram in Fig. 8 include the sudden onset of chaos and the jumps to non-local attractors.

5. Conclusion

Multibody systems with friction-affected bilateral constraints exist widely in engineering practice. This paper presents a general method for modeling and numerically simulating multibody systems of this type.

The modeling method based on constraint decomposition is presented. Each friction-affected bilateral constraint is decomposed into two constraints: the normal and tangential constraints. These two constraints can be further split into two unilateral constraints opposite to each other. Bilateral constraints are substituted with unilateral constraints, and constraint equations are substituted with complementarity inequalities, causing the absolute value of the Lagrange multipliers to disappear from the dynamic equations.

The LCP-based time-stepping method is adopted to avoid the exponential increase in the number of calculations with the growth in the number of friction-affected constraints. The benefit of time-stepping methods over event-driven integration methods is that no event-detection for stick-slip transition is needed when neglecting the difference between the static and kinetic friction coefficients.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant No. 10672007).

References