Numeration and type synthesis of 3-DOF orthogonal translational parallel manipulators

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Abstract

Low-degree-of-freedom (Low-DOF) parallel manipulators (PMs) have drawn extensive interests, particularly in type synthesis in which two main formal approaches were established by using the reciprocal screw system theory and Lie group theory. This paper aims at numeration and type synthesis of orthogonal translational parallel manipulators (OTPMs) by resorting to an integration of the group-based method and graphical representation of the topology. For this purpose, the concept of Cartesian DOF-characteristic matrix, originated from displacement subgroup and displacement submanifold, is proposed. A new approach based on the combination of the atlas of Cartesian DOF-characteristic matrix and the displacement group-theoretic method is addressed for both exhaustive classification and type synthesis of OTPMs. The proposed approach is prone to construct an orthogonal structure and is easy to realize the complete classification and exhaustive enumeration of this class of low-DOF PMs.

1. Introduction

Compared with conventional serial devices, parallel manipulators (PMs) have the merits of potential higher stiffness, lower inertia, larger payload/weight ratio, and better dynamic characteristics. But some disadvantages occur, especially in 6-DOF fully parallel manipulators. This is reflected in the complexity of the direct kinematics due to highly coupled position and orientation motions of the platform. In addition to this, manufacturing at a low cost and high accuracy is a challenge. However, fewer than 6 DOFs, or called low-DOF parallel manipulators [1], if properly designed, may alleviate or even overcome the above shortcomings. In particular, much effort has been dedicated to the design of 3-DOF translational PMs (TPMs) [2–9]. They received particular attention due to their vast industry applications as alternatives to traditional serial positioning systems, like as assembly manipulators, parallel kinematic machines (PKMs), and micromanipulators.

Although there are increasing designs for such a class of manipulators in the literatures, the mechanism types are still much fewer than those required. In this regards, a rigorous synthesis approach by the aid of mathematical tools appears appealing since it does address the difficulties encountered in the conceptual design of low-DOF PMs.

In the recent five years, there are increasing studies on the structural synthesis of low-DOF PMs in the literatures, also including TPMs. In particular, several systematic approaches have been proposed. They are the enumeration approach based on the general Chebyshev–Grübler–Kutzbach (CGK) mobility formula [10], the motion synthesis approach based on the general Chebyshev–Grübler–Kutzbach (CGK) mobility formula [10], the motion synthesis

Keywords: Type synthesis; Displacement group; Orthogonal translational parallel manipulator
approach based on the displacement subgroup theory [11–17] or single-open-chain theory [18,19], the constraint synthesis approach based on the reciprocal screw system theory [20–30], and the inference or conversion approach [31,32] based on the classical mechanism mechanism.

In a strict sense, the enumeration method belongs to number synthesis of mechanism and has its limitation [33]. Other than the motion synthesis method based on the displacement subgroups, which can describe the finite motion of kinematic pairs or kinematic chains, the constraint synthesis method based on the reciprocal screws generally falls into the category of instantaneous motion because both twists and wrenches are the elements of Lie algebra instead of smooth manifold. Whereas the method of inference originated from a known elemental or simple structure also proves effective in the construction of a new mechanism.

Although the above four approaches provide a systematic framework for the structural design of low-DOF PMs, the type synthesis of PMs with particular geometry which is required to fulfill some specified tasks, such as PMs with decouple motion [34,35] or the remote-center-of-motion (RCM) PMs [36], or type synthesis of PMs with particular kinematic features including isotropy [37–40], weak motion sensitivity [41], and high rotational ability [42] is still a comparably difficult task.

This paper mainly aims at exploring a simple and effective synthesis procedure for a class special decouple-motion TPMs, i.e., the orthogonal TPMs as a complement to the above general methods. For this purpose, the concept of Cartesian DOF-characteristic matrix (CDM), originated from the displacement subgroup and displacement submanifold, is proposed. A new approach based on the combination of the atlas of Cartesian DOF-characteristic matrix (ACDM) and the displacement group theory is addressed for both exhaustive enumerations and type synthesis of OTPMs. In order to verify the effectiveness of the proposed method, OTPMs with both symmetrical and asymmetrical architecture are synthesized accordingly.

2. Displacement subgroup and displacement submanifold

As found in Ref. [11], the set of 6-dimensional rigid motion can be endowed with the algebraic structure of a group, represented by $D$ as Lie group. Any further motion of a rigid body can be described by a subset of $D$, which may be either a group, called a displacement subgroup (DSG) or a displacement submanifold (DSM). The set of allowed relative displacements between two rigid bodies in a given kinematic chain is called kinematic bond. A kinematic chain generating the bond is named mechanical generator of the bond. In addition, Hervé enumerated all 12 kinds of displacement subgroups of $D$. Note that these subgroups can be represented in two configurations: nominal and conjugated configurations. For example, both $T(z)$ and $T(w)$ are the same representations of one-dimensional translational subgroup, but the former represents translations along the direction $z$ of Cartesian coordinate frame and the latter represents the movements in the direction of any axis $w$. In this paper, the descriptions of these subgroups in their nominal configurations are commonly used due to their special representations and potential advantages.

Amongst these displacement subgroups, $R(N, z)$, $T(z)$, $H(N, z, p)$, $C(N, z)$, $G(z)$, and $S(N)$ are associated with six lower kinematic pairs: Revolute pair (R), Prismatic pair (P), Helical pair (H), Cylinder pair (C), Planar pair (E) and Spherical pair (S). All these six lower pairs, combined with some composite joints such as Universal pair (U), $P_a$ and $U^*$ [43], can be regarded as primitive generators of DSGs or DSMs. Additional six displacement subgroups addressed in reference [11] include identity subgroup $e$, planar-translation subgroup $T_2(z)$, spatial translation subgroup $T$, translating-screw subgroup $Y(N, z, p)$, Schönflies subgroup $X(z)$, and Euclidean group $D$ itself. Hervé also enumerated the composition and intersection of different displacement subgroups. The intersection of subgroups follows the rules of intersection of sets.

For a serial kinematic chain or a serial manipulator composed of rigid bodies $1, 2, \ldots, n–1, n$, the allowed displacements of body $n$ relative to body $1$ (usually a fixed base) is a subset (DSG or DSM) of the group $D$, which is indeed a kinematic bond. This bond is generated by the composition by implementing the product of all displacement subgroups associated with the lower pairs in the kinematic chain. In a PM, however, the set of allowed rigid displacement of moving platform is obtained from the intersection of the kinematic bonds generated by all limb kinematic chains. Therefore, it is primarily necessary to recall some preliminaries of operations on DSGs/DSMs.

**Lemma 1.** Assuming that both $A$ and $B$ are displacement subgroups of $D$, the product of these two subgroups $A \cdot B$ is generally not a DSG, but a DSM included in $D$. At the same time, $A \cdot B$ is not commutative in most cases, i.e., $A \cdot B \neq B \cdot A$. On the contrary, if the product of these two DSGs can commute to each other, the resulting DSG reduces to an Abel group.

**Example 1.** Both $R(N, z)$ and $T(z)$ are displacement subgroups of $D$, and their product $R(N, z) \cdot T(z)$ can commute to each other, i.e.

$$R(N, z) \cdot T(z) = T(z) \cdot R(N, z)$$

Therefore, $R(N, z) \cdot T(z)$ constitutes an Abel DSG and is indeed a cylindrical-motion displacement subgroup $C(N, z)$.

**Lemma 2.** Assuming that both $A$ and $B$ are the displacement subgroups included in a common subgroup $Q$, i.e., $A \subseteq Q$, $B \subseteq Q$, due to the product closure in a subgroup, the product of these subsets $A \cdot B$ must be included in the same subgroup, i.e. $A \cdot B \subseteq Q$. 

Example 2. For two revolute pairs whose axes are parallel, as the mechanical generators of two subgroups $R(N_1, z)$ and $R(N_2, z)$, the serial arrangement of these two $R$ pairs produces a displacement subset $R(N_1, z) \cdot R(N_2, z)$. Note that both $R(N_1, z)$ and $R(N_2, z)$ are included in the subgroup $G(z)$, i.e., $R(N_1, z) \subset G(z)$, $R(N_2, z) \subset G(z)$. Due to the product closure in $G(z)$, the product of $R(N_1, z)$ and $R(N_2, z)$ is also included in $G(z)$, that is

$$R(N_1, z) \cdot R(N_2, z) \subset G(z)$$

(2)

In fact, $R(N_1, z) \cdot R(N_2, z)$ is a two-dimensional (2D) submanifold included in the 3D subgroup $G(z)$ but not a Lie subgroup of $G(z)$.

Corollary 2.1. The product of two same displacement subgroups is always equivalent to this subgroup.

Example 3. It is easy to derive the following equalities according to Lemma 2 and Corollary 2.1.

$$R(N, z) = R(N, z) \cdot R(N, z)$$

(3)

$$T(z) = T(z) \cdot T(z)$$

(4)

Corollary 2.2. Assuming that both $A$ and $B$ are included in a subgroup $Q$, i.e. $A \subseteq Q$, $B \subseteq Q$, and $\dim(A \cdot B) = \dim(Q)$, it can be deduced that the product $A \cdot B$ is equivalent to subgroup $Q$, i.e., $A \cdot B = Q$.

Example 4. Notice that $R(N, z) \subset G(z)$, and according to the product closure in a subgroup, it is deduced the following set equalities:

$$R(N_1, z) \cdot R(N_2, z) \cdot R(N_3, z) \subseteq G(z)$$

(5)

The product $R(N_1, z) \cdot R(N_2, z) \cdot R(N_3, z)$ is a three-dimensional (3D) submanifold included in $G(z)$, and $\dim(R(N_1, z) \cdot R(N_2, z) \cdot R(N_3, z)) = \dim(G(z)) = 3$, so according to Corollary 2.2 we have

$$R(N_1, z) \cdot R(N_2, z) \cdot R(N_3, z) = G(z)$$

(6)

Lemma 2 and its two corollaries play a key role in the displacement subgroup operation and leads to a series of equivalencies of kinematic chains. In some cases, however, the product of a DSM with another or a DSG or the product of three or more subgroups may not always be a DSM because of possible singularities.

Lemma 3. Assuming that both $A$ and $B$ are the displacement subgroups of $D$, the intersection of these two subgroups, represented by $A \cap B$, is always a subgroup. However, $A \cap B$ may be an improper subgroup containing only one element (the identity transform $E$). In this case, the subgroups are independent, or else they are group dependent.

Example 5.

$$R(N_1, z) \cap R(N_2, z) = \varepsilon$$

(7)

$$C(N, z) \cap S(N) = R(N, z)$$

(8)

3. Atlas of Cartesian DOF-characteristic matrix

A DOF-characteristic matrix, which may reflect the mobility characteristics of the moving platform (as a rigid body) with respect to the base in a PM or an end-effector of a serial manipulator, is denoted as

$$\begin{bmatrix}
\delta_{tx} & \delta_{ty} & \delta_{tz} \\
\delta_{rx} & \delta_{ry} & \delta_{rz}
\end{bmatrix}_{\#}$$

(9)

where the first row denotes translations of the body in the direction of $x, y, z$, the second row denotes 3D rotations in space, and "#$" represents rotation center point, if needed. As well known, one alternative method to describe the orientation of a rigid body is that its three rotation outputs can be also represented in Cartesian coordinate frame, thus Eq. (9) reduces to

$$\begin{bmatrix}
\delta_{tx} & \delta_{ty} & \delta_{tz} \\
\delta_{rx} & \delta_{ry} & \delta_{rz}
\end{bmatrix}_{\#}$$

(10)

Eq. (10) is defined as Cartesian DOF-characteristic matrix (CDM). Each element in Eq. (10) is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{Only motion existing} \\ 0 & \text{Only constraint existing} \end{cases}$$

(11)

For instance, the CDM of a 6-DOF serial manipulator, reflecting mobility characteristic of its end-effector, is

$$\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix},$$

and the CDM of a 3-DOF TPM corresponds to

$$\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix},$$

but the CDM of a 3-DOF spherical PM is denoted as

$$\begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}_N.$$

A CDM can be used not only for a reprehensive of the mobility characteristic of a manipulator but also as a symbolic description of a kinematic bond corresponding to a displacement subset of $D$. The displacement subset associated with typical rigid motions of manipulators and the corresponding CDMs are also enumerated in Table 1. Note that $R$ denotes rotation and $T$ denotes translation, while $xyz$ denotes the Cartesian coordinate frame with respect to three orthogonal rotations or orthogonal translations.

There may establish a mapping relationship between a DSG/DSM and a CDM associated with kinematic bonds (kinematic pairs or kinematic chains). Thus, operations of DSGs and DSMs may be mapped into those of CDMs, any an operation of CDMs involved here is called the Atlas of Cartesian DOF-characteristic Matrix (ACDM).

Since the DSG or DSM reflecting the output motion of the moving platform in a PM is written as the intersection of all subgroups or submanifolds generated by the limbs, the corresponding CDM of the platform can be also substituted into the intersection of all elements of CDMs related to the limbs. It is noted that this intersection operation (represented by symbol "$\cap$") should follow the rules of binary operation in a general case, and meet some geometric...
requirement as well. The resultant element is “1” only if every corresponding element in CDMs of all limbs is “1”. If one of them is “0”, the resultant element is “0”. For instance, for a TPM composed of three identical 3-DOF limbs, its corresponding ACDM is written as

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \cap \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \cap \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

(12)

The use of the intersection operation on CDMs, on the one hand, makes exhaustive enumeration of a class of PM simpler due to only a binary operation involved, on the other hand, the common constraint in this manipulator is easy to compute, i.e., the number of the common constraint in a manipulator equals to the sum of the corresponding “0” elements in the CDMs generated by all limbs. For instance, in Eq. (12), since the corresponding elements in the second row of CDMs generated by three limbs are all “0”, the number of the common constraint in this manipulator is thus 3. In addition, one can easily judge whether a PM is a non-over-constrained mechanism or not by using ACDM. If the sum of all “0” elements in the CDMs is equal to the number of the constraint applying in the PM, it is surely a non-over-constrained mechanism, and vice versa. For instance, in Eq. (11), the sum of all “0” elements in the CDMs is 9, but the constraint number corresponding to the PM is 3, which means that the PM is an overconstrained mechanism.

4. A type synthesis procedure for OTPMs

As discussed in some literatures, all orthogonal TPMs are indeed the decouple-motion ones and most of them can be deduced into the fully-isotropic mechanisms. Compared with a non-orthogonal TPM, they have a lot of outstanding advantages. First, the manipulator is economical in production and simple in calibration because of fully decouple motion in the direction of three translations of the manipulator. Second, the singularities in the workspace of the manipulator can be easily avoided by proper design because the singularities exist only at the edge of workspace. These advantages show that this kind of TPM has an extensive application potential in industry.

By resorting to a combination of the ACDM and DSG (DSM), a simple but effective procedure for the type synthesis of OTPMs is addressed as follows:

Step 1: Make a systematic classification and an exhaustive enumeration for the architectures of OTPMs by means of CDMs of both the moving platform and the limbs. For this purpose, decomposing CDM of the moving platform into the intersection of several CDMs corresponding to limbs is a primary importance. Thus, a complete ACDM reflecting limbs structure in the PM may be depicted.

Step 2: Find the mechanical generators corresponding to each possible limb in the desired PM by means of DSG/DSM operations.

Step 3: Construct a desired OTPM by selecting the architectures obtained in Step 1 and the mechanical generators found in Step 2. In addition, a reasonable arrangement of the limbs should be considered at the meantime.

From the next section on, we will apply this synthesis procedure to realize the type synthesis of the OTPMs.

5. Classification and exhaustive numerations of OTPMs

Generally, the limb in a PM can be classified as three categories, i.e., a proper-constraint limb, a deficient-con-
Table 2
Exhaustive numerations of OTPMs

<table>
<thead>
<tr>
<th>Type</th>
<th>$N_p$</th>
<th>$N_d$</th>
<th>$N_n$</th>
<th>$N_c$</th>
<th>ACDFM</th>
</tr>
</thead>
<tbody>
<tr>
<td>[B33]</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[B34]</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[B35]</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[B36]</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[B44]</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[B45]</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1 1 1 1 \cap 1 1 1 1</td>
</tr>
<tr>
<td>[T333]</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1 1 1 1 \cap 0 0 0 0 \cap 0 0 0 0</td>
</tr>
<tr>
<td>[T334]</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1 1 1 1 \cap 0 0 0 0 \cap 1 1 1 1</td>
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<tr>
<td>[T335]</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1 1 1 1 \cap 0 0 0 0 \cap 1 1 1 1</td>
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<tr>
<td>[T336]</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1 1 1 1 \cap 0 0 0 0 \cap 1 1 1 1</td>
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<tr>
<td>[T344]</td>
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<td>2</td>
<td>0</td>
<td>2</td>
<td>1 1 1 1 \cap 0 0 0 0 \cap 1 1 1 1</td>
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<tr>
<td>[T345]</td>
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<td>[T355]</td>
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<td>1</td>
<td>0 0 0 0 \cap 1 1 1 1 \cap 1 1 1 1</td>
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<td>[T346]</td>
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<tr>
<td>[T445]</td>
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<tr>
<td>[T455]</td>
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<td>0</td>
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<td>[T446]</td>
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<td>0</td>
<td>1 1 1 1 \cap 0 0 1 0 \cap 1 1 1 1</td>
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<tr>
<td>[T456]</td>
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<td>0</td>
<td>1 1 1 1 \cap 0 0 1 0 \cap 1 1 1 1</td>
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<tr>
<td>[T555]</td>
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<tr>
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<td>0</td>
<td>1 1 1 1 \cap 0 0 1 0 \cap 1 1 1 1</td>
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(continued on next page)
Table 2 (continued)

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<th>Type</th>
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<th>N_n</th>
<th>N_c</th>
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<td>[1 1 1]</td>
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</table>

6. Construction of OTPMs’ limbs

As found in Table 2, DSGs or DSMs generating all limbs of OTPMs contain four types: $T$, $X(z)$, $D_z$, and $D$.

In order to describe the limb structure explicitly, some symbols are defined first as follows: [ABC] denotes that the sequence of three DSGs/DSMs or three mechanical generators (A, B, C) is changeable one another. ABC denotes that the sequence of three DSGs/DSMs or three mechanical generators (A, B, C) is not changeable. In other words, there is only one arrangement for the limb, i.e., ABC.

6.1. $T$

The $T$ group has two types of proper Lie subgroups and two types of improper Lie subgroups. They are (a) $T(z)$, a set of rectilinear translations parallel to any given direction $z$ and (b) $T_2(z)$, a set of planar translations normal to given direction $z$. If and $T$ are two improper subgroups of $T$. Another, the $T$ group has two types of submanifolds. They are (a) $T_4(z)$, set of 1D linear motion along an open subset of the unit circle; (b) $T_2a(z)$, set of 2D linear motion along an open subset of the unit sphere.

Due to the product closure in the subgroup $T$, we can deduce a series of set equalities.

$$T = [T(x) \cdot T(y) \cdot T(z)] = [T(x) \cdot T(y) \cdot T_a(z)]$$

$$= [T(x) \cdot T_a(y) \cdot T_a(z)] = [T_a(x) \cdot T_a(y) \cdot T_a(z)]$$

$$= [T_2a(z) \cdot T_a(z)] = [T_2a(z) \cdot T_a(z)]$$

(13)

According to Eq. (13), the mechanical generators of $T$ can be obtained by placing in series of kinematic pairs generating subsets of $T$, as illustrated in Table 3.

6.2. $X(z)$

The 4-dimensional group $X(z)$ has at least seven types of proper Lie subgroups. They are $T(z)$, $T_2(z)$, $R(N, z)$, $H(N, z, p)$, $C(N, z)$, and $G(z)$.

All kinematic bonds of $G(z)$ and $C(N, z)$, and their corresponding mechanical generators have been enumerated in Refs. [12–16], here the result is given directly.

Due to the product closure in the subgroup $X(z)$, one can conclude the following set equalities denoting kinematic bonds and the corresponding mechanical generators of $X(z)$.

$$X(z) = [G(z) \cdot T(z)] = [G(z) \cdot T_a(z)] = [C(N, z) \cdot T_2(z)]$$

$$= [C(N, z) \cdot T_2a(z)] = [R(N, z) \cdot T]$$

(14)

Table 3

<table>
<thead>
<tr>
<th>Mechanical generators of $T$</th>
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<tbody>
<tr>
<td>[T(x) \cdot T(y) \cdot T(z)]</td>
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<tr>
<td>[T(x) \cdot T(y) \cdot T_4(z)]</td>
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<tr>
<td>[T_2(z) \cdot T(z)]</td>
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<tr>
<td>[T_2(z) \cdot T_4(z)]</td>
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<td>[T_2(z) \cdot T_2(z)]</td>
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<tr>
<td>[T_2(z) \cdot T_2a(z)]</td>
</tr>
<tr>
<td>[T_2a(z) \cdot T_2(z)]</td>
</tr>
</tbody>
</table>
Thus, the limb structures (equivalent to the mechanical
generators) corresponding to $X(z)$ can be derived from
Eq. (14), Tables 3 and 4, as listed in Table 5.

### 6.3. $D_2^3$

Due to the product closure in the displacement subgroup,
one can conclude that the following set equalities

\[
D_2^3(N, y, z) = T \cdot U(N, y, z) = [G_2(y) \cdot G(z)] = [G(y) \cdot G_2(z)]
\]

where $G_2(y)$ ($G_2(z)$) is a 2D submanifold included in the
group $G(y)$ ($G(z)$). The kinematic bond of $G_2(y)$ is obtained
by the removal of one translational subgroup of $G(y)$. Thus
all kinematic bonds of $G_2(y)$ and the corresponding mechanical
generators are enumerated in Table 6.

As known above, $G_2(y) \cdot G(z)$ is a kinematic bond of
$D_2^3(N, y, z)$, and

\[
G_2(y) \cdot G(z) = R(N_1, y) \cdot R(N_2, y) \cdot G(z)
\]

According to the set intersection operation, we have

\[
G_2(y) \cdot G(z) = R(N_1, y) \cdot R(N_2, y) \cdot G(z) \cap T = T
\] (17)

Multiplying the two sides of Eq. (17) by $R(N_1, y)$ on the
right-hand side, we have

\[
R(N_1, y) \cdot R(N_2, y) \cdot G(z) \cdot R(N_1, y) \cap T \cdot R(N_1, y) = T \cdot R(N_1, y)
\] (18)

The commutation in the product of $R(N_1, y)$ and $T$
yields

\[
R(N_1, y) \cdot R(N_2, y) \cdot G(z) \cdot R(N_1, y) \cap R(N_1, y) \cdot T = R(N_1, y) \cdot T
\] (19)

The common factor $R(N_1, y)$ at both sides of Eq. (19)
can be eliminated, thus we have

\[
R(N_2, y) \cdot G(z) \cdot R(N_1, y) \cap T = T
\] (20)

Hence, $R(N_2, y) \cdot G(z) \cdot R(N_1, y)$ can be also regarded as
a kinematic bond of $D_2^3(N, y, z)$. Its corresponding mechanical
generators include $[^1R \cdot]^2R \cdot ^3R$, $[^1R \cdot]^2R \cdot ^3R$, etc.

In a similar way, we can also derive that other kinematic
bonds of $D_2^3(N, y, z)$.

\[
R(N_3, z) \cdot R(M_1, y) \cdot R(M_2, y) \cdot R(N_1, z) \cdot R(N_2, z)
\]

\[
\times ([^1R \cdot]^2R \cdot ^3R)
\]

\[
T(x) \cdot R(M_1, y) \cdot R(M_2, y) \cdot R(N_1, z) \cdot R(N_2, z) ([^1R \cdot]^2R \cdot ^3R)
\]

\[
R(N_3, z) \cdot R(M_1, y) \cdot R(M_2, y) \cdot [R(N_1, z) \cdot T(x)]
\]

\[
\times ([^1R \cdot]^2R \cdot ^3R)
\]

Thus, three groups of mechanical generators corresponding
to $D_2^3(N, y, z)$ can be derived from above equalities,
and are listed in Table 7.
7. Construction of OTPMs

Based on the previous results on classifications of the OTPMs, and the derived limb structures, it is not difficult to construct a general orthogonal TPM.

As well known, for a 3-legged OPM (generally called a tripod platform), there are usually three typical layouts for the limbs from the view of axial or linear symmetry, as illustrated in Fig. 1.

7.1. Construction of the symmetric OTPMs composed of limbs generating $X(z)$

The symmetric OTPMs composed of limbs generating $X(z)$ can be constructed by means of the resulting limb structures in Table 5. In this case, it corresponds to two classes of TPMs, i.e., [T444] and [B44].

First, taking construction of an OTPM composed of three limbs generating $X$ as an example, the whole procedure for construction of this OTPM is given as follows.

There are two different ACDMs corresponding to this type of OTPM, and these can be written as

**Case 1:**

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (21)$$

**Case 2:**

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (22)$$

For case 1, three limb structures corresponding to three CDMs on the right-hand side of Eq. (21) can be selected as $^{\chi}_{P}P^{*}R$, $^{\chi}_{C}P^{*}R$, and $^{\chi}_{C}P^{*}R$ according to Table 5. Thus we can construct an axis-symmetry OTPM, as illustrated in Fig. 2(a).

**Fig. 1.** Typical layouts of limbs in an orthogonal tripod platform. (a) Axial symmetry; (b) linear symmetry; (c) linear symmetry.
For case 2, three limb structures corresponding to two different CDMs can be selected as \( x_{C'y_Px_R}, x_{C'y_Px_R}, \) and \( z_{C'x_Pz_R} \) according to Table 5. Thus we can construct a line-symmetry OTPM, as illustrated in Fig. 2(b).

Using the same method, we can construct an axis-symmetry 3-RPRR OTPM and a line-symmetry 3-RP\(_a\)RR OTPM, as illustrated in Fig. 3.

### 7.2. Construction of the symmetric OTPMs composed of limbs generating \( D_3^1 \)

The symmetric TPMs composed of limbs generating \( D_3^1 \) can be constructed according to the resulting limb structures from the Table 7. In this case, it corresponds to the class \([T555]\).

Taking an 3-RCC OTPM composed of limbs generating \( D_3^1 \) as an example, the whole procedure for the construction of this OTPM is given as follows. The corresponding ACDM can be written as

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix} \cap \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix} \cap \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

(23)

where three limb structures corresponding to three CDMs can be selected as \( ^R\)R\(^C\)C, \( ^R\)R\(^C\)C, and \( ^R\)R\(^C\)C according to the result in the Table 7. Thus, we can construct an axis-symmetry OTPM, as illustrated in Fig. 4.

### 7.3. Construction of the asymmetrical OTPMs

From Table 2, asymmetrical TPMs include many subtypes. In this paper, taking classes \([B44],[B45],[T445]\) and \([T455]\) for instances, the whole procedure for construction of the corresponding OTPMs is described as follows.

Since the ACDM corresponding to the class \([B44]\) contains two different CDMs, the corresponding limb structures may be \( ^R\)P\(_a\)^R\(^C\) and \( ^R\)R\(^C\), respectively. Thus, using the method described above, we can construct an OTPM belonging to the class \([B44]\), as illustrated in Fig. 5.
7.4. Construction of the OTPMs containing one and more non-constraint limb

In the family of the OTPMs, although the structural architecture of the whole mechanism is symmetrical, there exist some special classes, such as classes [T336], [T366], [T446], [Q3336], [Q3666], [Q4446], [Q5556], [P33666], [P44666], and [H333666], which characterize at least one non-constraint limb in the TPM. Amongst all these 10 classes, the class [Q3666] is the most familiar as it is usually found in the design of PKM. In order to increase the stiffness of PKM, we can also prefer classes [P33666], [P44666], and [H333666] to [Q3666] since there are more limbs between the fixed platform and the moving platform.

Since the ACDM corresponding to the class [Q4446] contains two different CDMs, the corresponding limb structures can be selected as RPR and SPS, respectively. Thus using the method described before, we can construct an OTPM belonging to the class [Q4446], as illustrated in Fig. 6(a). Similarly, another OTPMs belonging to the class [Q5556] can also be constructed, as illustrated in Fig. 6(b).

8. Conclusions

The concept of CDM and its operation mode, taken birth from properties of displacement subgroup and displacement submanifold, are proposed in this paper. Then a new approach based on ACDM, and matching with the synthesis method of Lie group effectively, is addressed for both the classification and the type synthesis OTPMs.
The proposed approach has some outstanding advantages. First, only very simple binary operation on CDM is involved. Second, it is very intuitive and especially simple to construct an orthogonal structure due to the properties of CDM. Third, it becomes easy to realize the complete classification and exhaustive enumeration of the OTPMs by means of ACM. In order to verify the effectiveness of the proposed method, type synthesis of OTPMs is performed, which can enrich the family of OTPMs extensively.

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References


