An eigenfunction expansion method for the elastodynamic response of an elastic solid with mixed boundary surfaces

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Abstract

The method of eigenfunction expansion is one of the most elegant methods for solving elastodynamic problems. The solution obtained from it is more concise than that obtained from the integral transform technique. Traditional eigenfunction expansion method is used for the elastodynamic problems with displacement and traction boundary conditions. In this paper, the method is generalized to study the elastodynamic response of an elastic solid with mixed boundary surfaces, and the exact analytical solution is derived. The dynamic response of a finite-length solid aluminum cylinder with two mixed end boundaries is numerically evaluated. The result computed from the analytical solution agrees very well with that obtained from finite element method (FEM).

Keywords: Eigenfunction expansion; Elastodynamic response; Mixed boundary

1. Introduction

Generally, two fundamental methods, the integral transform and eigenfunction expansion techniques, are employed to study the elastodynamic responses of elastic solids. The eigenfunction expansion method for elastodynamics owes its origin to the Williams type modal solution technique, which has been used to study the dynamic responses of beams and shells [1,2]. Reissmann extended this method to the realm of three-dimensional elastodynamics [3]. Eringen and Suhubi discussed the method in detail on the basis of the work of Reissmann [4]. Compared with the integral transform method, the eigenfunction expansion method is more suitable to study the excitation of the waves in elastic waveguides. The solution obtained from the latter is more concise than that obtained from the former. Thus, Pao thought that the eigenfunction expansion method is one of the most elegant methods for solving elastodynamic problems [5]. Liu and Qu [6] studied the transient wave propagation in a circular annulus subject to transient excitation on its outer surface by this method. Tang and Cheng [7] used the method to investigate the laser-generated guided elastic waves in an infinite hollow cylinder.

The eigenfunction expansion method used by Reissmann [3], Eringen and Suhubi [4] is only for the elastodynamic responses of the elastic solids including displacement and traction boundary surfaces. In this study, the method is generalized to solve the elastodynamic problems including mixed boundary conditions. It should be pointed out that the boundary B of an elastic solid is called mixed in some published papers and books, if all the displacement components $u_i (i = 1, 2, 3)$ on the boundary $B_1$ and $\sigma_{ij}n_j (j = 1, 2, 3)$ on the boundary $B_2$ are prescribed, where $\sigma_{ij}$ is the stress...
component, \( n_i \) is the component of the unit normal vector of surface \( B_2 \) and \( B_1 \cup B_2 = B \). But in this study, the mixed boundary refers to a surface on which \( \sigma_{ij}(t) (i, j = 1, 2) \) and \( u_i \) are prescribed, for example, the rigid-smooth boundary. And this kind of definition has been used by Miklowitz [8].

2. Elastodynamic responses of elastic solids by eigenfunction expansions

A well-posed mixed boundary-initial value problem of elastodynamics in a region \( V \) bounded by \( B \) is formulated to determine the solution of Navier’s equation

\[
(\ddot{\gamma} + 2\mu \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} + \rho \mathbf{f}(x, t)) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad \text{in } V
\]

under the boundary condition

\[
\mathbf{u}(x, t) = \mathbf{h}(x, t) \quad \text{on } B_1
\]

\[
\sigma_{(i)}(x, t) = p(x, t) \quad \text{on } B_2
\]

\[
\sigma^{(1)}_{(i)}(x, t) = \xi(x, t) \quad \text{on } B_3
\]

\[
\sigma^{(2)}_{(i)}(x, t) = \eta(x, t) \quad \text{on } B_3
\]

\[
u_3(x, t) = \zeta(x, t) \quad \text{on } B_3
\]

and the initial conditions

\[
\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{v}(x, 0) = v_0(x)
\]

in \( V \) where \( \gamma \) and \( \mu \) are the Lamé constants of the material; \( \rho \) is the density and it is a constant for a homogeneous elastic solid; \( \mathbf{u}(x, t) \) and \( \mathbf{v}(x, t) \) are the displacement and velocity field vectors, respectively; \( \sigma_{(i)} \) is the \( i \)th component of \( \sigma_{(i)}(x, t) \); and \( B_1 \cup B_2 \cup B_3 = B \).

Firstly, we consider the eigenvalue problem corresponding to (1)–(7), namely,

\[
(\ddot{\gamma} + 2\mu \nabla \cdot \psi(x) - \mu \nabla \times \nabla \times \psi(x) + \rho \omega^2 \psi(x)) = 0
\]

in \( V \) where \( \psi(x) \) satisfies the following homogeneous boundary conditions

\[
\psi(x) = 0 \quad \text{on } B_1
\]

\[
\sigma_{(i)}(x) = 0 \quad \text{on } B_2
\]

\[
\sigma^{(1)}_{(i)}(x) = 0 \quad \text{on } B_3
\]

\[
\sigma^{(2)}_{(i)}(x) = 0 \quad \text{on } B_3
\]

\[
u_3(x) = 0 \quad \text{on } B_3
\]

The solution of (8)–(13) leads to an infinite sequence for the eigenvalues \( \omega_n (n = 1, 2, \ldots) \). A particular mode of oscillation \( \psi_n(x) \) associated with the eigenvalue \( \omega_n \) is called eigenfunction [4]. It is easy to prove that all eigenvalues \( \omega_n^2 \) of (8)–(13) are real and non-negative. The orthogonality of the eigenfunctions \( \psi_n(x) \) corresponding to homogeneous displacement and traction boundary conditions can be proved by using the reciprocal theorem. Note that the proof presented by Eringen and Suhubi was based on the assumption that eigenvalues \( \omega_n \) are non-normal [4], i.e. only one eigenfunction is associated with a given eigenvalue \( \omega_n \). Actually, the eigenvalues \( \omega_n \) corresponding to the free vibrations of most elastic waveguides are normal, i.e. multi-eigenfunctions are associated with the given eigenvalue \( \omega_n \) [9]. Generally, the proof about the orthogonality of the eigenfunctions is more complex than that presented by Eringen and Suhubi [4] when the eigenvalues \( \omega_n \) are normal. If the eigenvalues corresponding to (8)–(13) are non-normal, the orthogonality of the eigenfunctions can also be easily proved by the reciprocal theorem, and we have

\[
\int_V \rho \psi_n(x) \cdot \psi_m(x) \, dV = \begin{cases} 0, & n \neq m \\ M_m, & n = m \end{cases}
\]

where \( M_m \) is the norm of \( \psi_m(x) \).

Secondly, we try to find a quasistatic solution of the following equation

\[
(\ddot{\gamma} + 2\mu \nabla \cdot \varphi(x, t) - \mu \nabla \times \nabla \times \varphi(x, t) + \rho \mathbf{f}(x, t)) = 0
\]

in \( V \) under the boundary conditions

\[
\varphi(x, t) = \mathbf{h}(x, t) \quad \text{on } B_1
\]

\[
\sigma_{(i)}(x) = p(x, t) \quad \text{on } B_2
\]

\[
\sigma^{(1)}_{(i)}(x) = \xi(x, t) \quad \text{on } B_3
\]

\[
\sigma^{(2)}_{(i)}(x) = \eta(x, t) \quad \text{on } B_3
\]

\[
\psi_3(x, t) = \zeta(x, t) \quad \text{on } B_3
\]

Although the solution of (15)–(20) may be difficult to find, we will show that the explicit solution \( \varphi(x, t) \) is not necessary for the final formulation.

Now, we presume that the general solution of (1) under the boundary conditions (2)–(6) and the initial condition (7) can be represented by

\[
\mathbf{u}(x, t) = \varphi(x, t) + \sum_{n=1}^{\infty} \phi_n(t) \psi_n x
\]

where \( \varphi(x, t) \) is the solution of (15)–(20), \( \psi_n(x) \) are the eigenfunctions, i.e. the solutions of (8)–(13), and \( \phi_n(t) \) are the unknown functions of time only. Substituting (21) into (1) and employing (15), we obtain

\[
\sum_{n=1}^{\infty} \left( \phi_n + \omega_n^2 \phi_n \right) \psi_n(x) = -\varphi(x, t)
\]

Both sides of (22) are multiplied by \( \rho \psi_n(x) \) and then integrated over \( V \). Invoking (14), we have

\[
\dot{\phi}_m + \omega_m^2 \phi_m = \dot{\phi}_m(t)
\]

where

\[
\phi_m(t) = -\frac{1}{M_m} \int_V \rho \varphi(x, t) \cdot \psi_m(x) \, dV
\]
It can be derived from (23) by using Laplace integral transform technique that
\[
\phi_m(t) = \alpha_m \cos \omega_m t + \frac{1}{\omega_m} \beta_m \sin \omega_m t + \Phi_m(t) - \omega_m \int_0^t \Phi_m(\tau) \sin \omega_m (t - \tau) d\tau
\]
where
\[
\alpha_m = \phi_m(0) - \Phi_m(0) = \frac{1}{M_m} \int _v \rho \mathbf{u}_0(x) \cdot \mathbf{\psi}_m(x) \, dV
\]
and
\[
\beta_m = \dot{\phi}_m(0) - \dot{\Phi}_m(0) = \frac{1}{M_m} \int _v \rho \mathbf{v}_0(x) \cdot \mathbf{\psi}_m(x) \, dV
\]
Substituting (25) into (21), we obtain
\[
\mathbf{u}(x, t) = \mathbf{\varphi}(x, t) + \sum_{m=1}^{\infty} \Phi_m(t) \mathbf{\psi}_m(x)
\]

If the eigenfunctions \( \mathbf{\psi}_m(x) \) constitute a complete orthogonal set, we can expand \( \mathbf{\varphi}(x, t) \) as
\[
\mathbf{\varphi}(x, t) = \sum_{m=1}^{\infty} g_m(t) \mathbf{\psi}_m(x)
\]
where
\[
g_m(t) = \frac{1}{M_m} \int _\Omega \rho \mathbf{\varphi}(x, t) \cdot \mathbf{\psi}_m(x) \, dV
\]
Comparing (30) with (24), we find that
\[
g_m(t) = -\Phi_m(t)
\]
Substituting (31) into (29), we can obtain that
\[
\mathbf{\varphi}(x, t) = -\sum_{m=1}^{\infty} \Phi_m(t) \mathbf{\psi}_m(x)
\]
Substitution of the above expression into (28) yields
\[
\mathbf{u}(x, t) = \sum_{m=1}^{\infty} \left[ \alpha_m \cos \omega_m t + \frac{1}{\omega_m} \beta_m \sin \omega_m t - \omega_m \int_0^t \Phi_m(\tau) \sin \omega_m (t - \tau) d\tau \right] \mathbf{\psi}_m(x)
\]
According to (8), we have
\[
\mathbf{\psi}_m(x) = -\frac{1}{\rho \omega_m^2} \mathbf{L}(\mathbf{\psi}_m)
\]
where the operator \( \mathbf{L} \) is defined as \( \mathbf{L} = (\lambda + 2\mu) \nabla \nabla - \mu \nabla \times \nabla \times \). Substituting (34) into (24), we obtain
\[
\Phi_m(t) = \frac{1}{M_m \omega_m^2} \int _v [\mathbf{\varphi}(x, t) \cdot \mathbf{L}(\mathbf{\psi}_m)] \, dV
\]
It is clear that
\[
\int_v [\mathbf{\varphi}(x, t) \cdot \mathbf{L}(\mathbf{\psi}_m) - \mathbf{\psi}_m \cdot \mathbf{L}(\mathbf{\varphi}(x, t))] \, dV
\]
\[
= \int_v \left[ \frac{\partial \mathbf{\sigma}_j(\mathbf{\psi}_m)}{\partial \mathbf{x}_j} - \frac{\partial \mathbf{\varphi}}{\partial \mathbf{x}_j} \right] \, dV
\]
Then
\[
\int_v [\mathbf{\varphi}(x, t) \cdot \mathbf{L}(\mathbf{\psi}_m)] \, dV = \int_v [\mathbf{\psi}_m \cdot \mathbf{L}(\mathbf{\varphi}(x, t))] \, dV
\]
\[
+ \int_v \left[ \frac{\partial \mathbf{\sigma}_j(\mathbf{\psi}_m)}{\partial \mathbf{x}_j} - \frac{\partial \mathbf{\varphi}}{\partial \mathbf{x}_j} \right] \, dV
\]
There exists
\[
\int_v \left[ \frac{\partial \mathbf{\varphi}}{\partial \mathbf{x}_j} \cdot \mathbf{\sigma}_j(\mathbf{\psi}_m) - \frac{\partial \mathbf{\psi}_m(\phi)}{\partial \mathbf{x}_j} \mathbf{\sigma}_j(\phi) \right] \, dV = 0
\]
The detailed proof about it is presented in Appendix. Employing the divergence theorem and (38), we have
\[
\int_v \left[ \frac{\partial \mathbf{\sigma}_j(\mathbf{\psi}_m)}{\partial \mathbf{x}_j} - \frac{\partial \mathbf{\psi}_m(\phi)}{\partial \mathbf{x}_j} \mathbf{\sigma}_j(\phi) \right] \, dV
\]
\[
+ \int _B \left[ \mathbf{\psi}_m \cdot \mathbf{\sigma}(\mathbf{\psi}_m) - \mathbf{\psi}_m \cdot \mathbf{\sigma}(\phi) \right] \, dS = 0
\]
Substitution of (39) into (37) leads to
\[
\int _v \left[ \mathbf{\psi}_m \cdot \mathbf{L}(\mathbf{\varphi}(x, t)) \right] \, dV
\]
\[
= \int_v \left[ \mathbf{\psi}_m \cdot \mathbf{L}(\mathbf{\varphi}(x, t)) \right] \, dV
\]
\[
+ \int _B \left[ \mathbf{\psi}_m \cdot \mathbf{\sigma}(\mathbf{\psi}_m) - \mathbf{\psi}_m \cdot \mathbf{\sigma}(\phi) \right] \, dS
\]
Upon application of (15), we obtain
\[
\int_v [\mathbf{\psi}_m \cdot \mathbf{L}(\mathbf{\varphi}(x, t))] \, dV = -\int_v \rho \mathbf{\varphi} \cdot \mathbf{f}(x, t) \, dV
\]
Apparently,
\[
\int _B \left[ \mathbf{\psi}_m \cdot \mathbf{\sigma}(\phi) \right] \, dS
\]
\[
= \int _{B_1 + B_2 + B_3} \left[ \mathbf{\varphi}(x, t) \cdot \mathbf{\sigma}(\phi) \right] \, dS
\]
We obtain after employing (9) and (16) that
\[
\int _{B_1} [\mathbf{\varphi}(x, t) \cdot \mathbf{\sigma}(\phi) - \mathbf{\psi}_m \cdot \mathbf{\sigma}(\phi)] \, dS
\]
\[
= \int _{B_1} \mathbf{h}(x, t) \cdot \mathbf{\sigma}(\phi) \, dS
\]
Employing (10) and (17), we have
\[
\int _{B_2} [\mathbf{\varphi}(x, t) \cdot \mathbf{\sigma}(\phi) - \mathbf{\psi}_m \cdot \mathbf{\sigma}(\phi)] \, dS
\]
\[
= -\int _{B_2} \mathbf{\psi}_m \cdot \mathbf{p}(x, t) \, dS
\]
And employing (11)–(13) and (18)–(20), we obtain
\[
\int_{B_1} [\phi(x,t) \cdot \sigma_{(n)}(\psi_m) - \psi_m \cdot \sigma_{(n)}(\varphi)]dS + \int_{B_1} [\xi(x,t) \sigma_{(n)}^{(3)}(\psi_m) - \psi_m^{(1)} \xi(x,t) - \psi_m^{(2)} \eta(x,t)]dS = 0
\]
(45)
Substituting (41)–(45) into (40), and then into (35), we have
\[
\Phi_m(t) = \frac{1}{M_m\omega_m} \left\{ -\int_V \rho \psi_m \cdot f(x,t) dV + \int_{B_1} h(x,t) \cdot \sigma_{(n)}(\psi_m) dS - \int_{B_1} \psi_m \cdot p(x,t) dS \right. \\
\left. + \int_{B_1} \left[ \xi(x,t) \sigma_{(n)}^{(3)}(\psi_m) - \psi_m^{(1)} \xi(x,t) - \psi_m^{(2)} \eta(x,t) \right] dS \right\}
\]
(46)
Expressions (33), (26), (27), (14) and (46) are the formal solution of (1) under the boundary conditions (2)–(6) and the initial condition (7).

3. Examples

Fig. 1 shows a solid aluminum cylinder with length 2l and radius a. Two end boundary conditions of it are \( \sigma_{(n)}^{(1)}(x,t) = \xi(x,t) = 0, \sigma_{(n)}^{(2)}(x,t) = \eta(x,t) = 0 \) and \( u_i(x,t) = \zeta(x,t) \neq 0 \). The body force \( f(r, \theta, z, t) \) is axisymmetric and axial, as shown in Fig. 1. The surface force \( p(a, \theta, z, t) \) is applied on the lateral surface axisymmetrically and radially. And we set
\[
f(r, \theta, z, t) = AT(t) e_r, \quad 0 \leq r \leq a/3, \quad 0 < \theta \leq 2\pi, \\
d/2 \leq z \leq d/2
\]
(47)
\[
p(a, \theta, z, t) = BT(t) e_r, \quad 0 < \theta \leq 2\pi, \quad -d/2 \leq z \leq d/2
\]
(48)
\[
\zeta(r, \theta, z, t) = CT(t), \quad 0 \leq r \leq a, \quad 0 < \theta \leq 2\pi, \quad z = \pm l
\]
(49)
where \( A = 3.0 \times 10^5, B = 3.704 \times 10^4, \) and \( C = 1.0 \times 10^{-8} \). The units of \( f, p \) and \( \zeta \) are N/kg, N/m² and m, respectively. And it is assumed that
\[
u_0(x) = 0, \quad v_0(x) = 0
\]
(50)
In this section, we will employ the above theory to study the elastodynamic response of the solid cylinder. The material parameters of the aluminum cylinder are \( \rho = 2.7 \times 10^3 \) kg/m³, \( \mu = 25 \) GPa and the Poisson coefficient \( \gamma = 0.35 \). The geometrical parameters of it are \( a = 0.015 \) m and \( l = 2.0 \) m. The value of \( d \) in Eqs. (47) and (48) is 0.004 m. Clearly, only longitudinal wave modes can be generated. Furthermore, \( T(t) \) is set to consist of 6 cycles, 50 kHz tonebursts modulated by a Hanning window. Then we can know from the dispersion curves of longitudinal waves that only mode \( L(0,1) \) is excited in this case. Substituting (50) into (26), (27), and (47)–(49) into (46), then employing (33), and noting that no displacement boundary condition exists in this problem, we obtain
\[
u(r, z, t) = \sum_n \left\{ \frac{2\pi}{M_m\omega_m} \int_0^\alpha \left[ \int_0^\Gamma \int_0^a A\rho \psi_m^{(2)}(r, z) T(t) r dr dz \right. \\
+ a \int_0^\Gamma \int_0^a B\psi_m^{(1)}(a, z) T(t) z - \int_0^a CT(t) \sigma_{(n)}^{(3)}(\psi_m)_{l=1} r dr \left. - \int_0^a CT(t) \sigma_{(n)}^{(2)}(\psi_m)_{l=1} r dr \right] \sin \omega_m(t - \tau) d\tau \right\} \psi_m(r, z)
\]
(51)
where
\[
\sigma_{(n)}^{(3)}(\psi_m) = \begin{cases} -[\lambda \nabla \cdot \psi_m + 2\mu \psi_m^{(2)}], & z = -l \\
[\lambda \nabla \cdot \psi_m + 2\mu \psi_m^{(2)}], & z = l 
\end{cases}
\]
(52)
Fig. 2 shows the transient waveforms computed from Eqs. (51), (52) at \( z = 0.8 \) m, where Fig. 2(a) and (b) are the waves excited by the body force \( f(r, \theta, z, t) \) and surface force \( p(a, \theta, z, t) \), respectively. Fig. 2(c) shows the propagation of \( \zeta(r, \theta, \pm l, t) \), and Fig. 2(d) is the total transient waveform. Fig. 3 gives the transient waveforms simulated with finite element method (FEM) at the same point. The explicit algorithm is employed in the FEM simulation. Comparing Fig. 3(a)–(d) with Fig. 2(a)–(d), we can find that they agree very well.

4. Conclusions

The method of eigenfunction expansion is generalized to study the elastodynamic response of the elastic solids including displacement boundary, traction boundary and...
here, the dynamic response of a finite-length solid cylinder with two mixed end boundaries is studied. The results computed from the analytical solution agree very well with those simulated by FEM.

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Appendix A

Proving (37), i.e.

\[
\int_V \left[ \frac{\partial \psi_j}{\partial x_j} \cdot \sigma_{ij}(\psi) - \frac{\partial \psi_j}{\partial x_j} \cdot \sigma_{ij}(\varphi) \right] dV = 0
\]

is equivalent to proving

\[
\int_V \left[ \frac{\partial u_j}{\partial x_j} \cdot \sigma_{ij}(\varphi) - \frac{\partial v_j}{\partial x_j} \cdot \sigma_{ij}(\varphi) \right] dV = 0 \tag{A1}
\]

But the latter is more readable than the former.

Proof. The first term of the integrand can be rewritten as

\[
\frac{\partial u_j}{\partial x_j} \cdot \sigma_{ij}(\varphi) = \frac{\partial u_1}{\partial x_1} \sigma_{11}(\varphi) + \frac{\partial u_2}{\partial x_2} \sigma_{22}(\varphi) + \frac{\partial u_3}{\partial x_3} \sigma_{33}(\varphi)
\]

We have

\[
\frac{\partial u_1}{\partial x_1} \sigma_{11}(\varphi) + \frac{\partial u_2}{\partial x_2} \sigma_{22}(\varphi) + \frac{\partial u_3}{\partial x_3} \sigma_{33}(\varphi)
\]

\[
= \frac{\partial u_1}{\partial x_1} \left( \lambda \nabla \cdot v + 2 \mu \frac{\partial v_1}{\partial x_1} \right) + \frac{\partial u_2}{\partial x_2} \left( \lambda \nabla \cdot v + 2 \mu \frac{\partial v_2}{\partial x_2} \right) + \frac{\partial u_3}{\partial x_3} \left( \lambda \nabla \cdot v + 2 \mu \frac{\partial v_3}{\partial x_3} \right)
\]

\[
= \lambda \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \nabla \cdot u
\]

mixed boundary surfaces. The exact analytical solution obtained with this method is concise, based on which it is easy to analyze the influences of the external forces on the excitation. In order to illustrate the theory presented

Fig. 2. The transient waveforms simulated with the method of eigenfunction expansion. (a) Excited by \( f(r, \theta, z, t) \); (b) excited by \( p(\alpha, \theta, z, t) \); (c) propagation of \( \zeta(r, \theta, z, t) \); (d) total waveform.

Fig. 3. The transient waveforms simulated with FEM. (a) Excited by \( f(r, \theta, z, t) \); (b) excited by \( p(\alpha, \theta, z, t) \); (c) propagation of \( \zeta(r, \theta, z, t) \); (d) total waveform.
Substituting (A4) and (A3) into (A2), we obtain

\[
\frac{\partial u_1}{\partial x_2} \sigma_{12}(v) + \frac{\partial u_1}{\partial x_3} \sigma_{13}(v) + \frac{\partial u_2}{\partial x_1} \sigma_{21}(v) + \frac{\partial u_2}{\partial x_3} \sigma_{23}(v) + \frac{\partial u_3}{\partial x_1} \sigma_{31}(v)
\]

\[+ \frac{\partial u_3}{\partial x_2} \sigma_{32}(v) = \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + \mu \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right)
\]

\[+ \mu \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right)
\]

\[= \sigma_{12}(u) \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + \sigma_{13}(u) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right)
\]

\[+ \sigma_{23}(u) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) = \sigma_{12}(u) + \sigma_{23}(u) + \sigma_{13}(u)
\]

\[+ \frac{\partial v_3}{\partial x_2} \sigma_{31}(u) + \frac{\partial v_2}{\partial x_3} \sigma_{25}(u)
\]

\[+ \frac{\partial v_1}{\partial x_2} \sigma_{23}(u) + \frac{\partial v_3}{\partial x_2} \sigma_{32}(u) + \frac{\partial v_2}{\partial x_2} \sigma_{23}(u)
\]

\[= \frac{\partial v_j}{\partial x_j} \cdot \sigma_{ij}(u)
\]

(A4)

Substituting (A4) and (A3) into (A2), we obtain

\[
\int_V \left[ \frac{\partial u_i}{\partial x_j} \cdot \sigma_{ij}(v) - \frac{\partial v_i}{\partial x_j} \cdot \sigma_{ij}(u) \right] dV = 0
\]

(A6)

This completes the proof. □

References